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Recursion operators and bi-Hamiltonian formulations of the Landau–Lifshitz equation hierarchies

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Abstract
Using a polynomial pencil of Lax pairs we give a new bi-Hamiltonian formulation for the Landau–Lifshitz equation hierarchy and find a new recursion operator related to it.

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1. Introduction

In the past decades a number of infinite-dimensional nonlinear dynamic equations (systems) have been discovered which attracted great interest, the so-called soliton equations, such as the Korteweg-de Vries equation, the nonlinear Schrödinger equation, the sin-Gordon equation, etc, see [5, 6]. Their characteristic property is that they can be regarded as the compatibility condition for pairs of linear operators (L-A pairs or Lax pairs). As a result from the above representation as compatibility condition (Lax representation) it has been discovered that the soliton equations are completely integrable Hamiltonian systems. They always appear in hierarchies, have infinite series of conservation laws, etc [6]. But probably the most important fact regarding the soliton equations is that new techniques for obtaining large families of exact solutions has been developed, based on the Lax representations for these equations. So nowadays the simplest soliton equations (usually the first nonlinear equations that appear in the corresponding hierarchies) are famous and considerable literature is dedicated to each of them. Most of the soliton equations also have important physical applications and the first interest to them was almost entirely motivated by it, but today the mathematical importance of the soliton equations sometimes eclipses the physical one, since the possibility of obtaining nontrivial exact solutions for nonlinear partial differential equations is a very rare phenomenon.

One of the famous soliton equations is the so-called Landau–Lifshitz equation (LLeq):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx} + \mathbf{S} \times R(\mathbf{S}), \qquad \lim_{x\to\pm\infty} \mathbf{S}(x,t) = (0,0,1), \tag{1}$$

where $\mathbf{S}(x, t)$ is a three-dimensional vector field, depending on one spatial variable x and the time t, taking its values on the two-dimensional unit sphere; $R = \mathrm{diag}(r_1, r_2, r_3)$ is a constant diagonal matrix with diagonal elements $r_i \geqslant 0$, and the vector field $R(\mathbf{S})$ has components $r_i S_i$. The LL equation is a classical limit of a quantum system describing a ferromagnetic chain in which the nth atom has a spin s_n and interacts only with the nth and $n+1$th neighbour, see [13]. As a soliton equation it is famous because in its study there has been applied for the first time the L-A pair which was elliptic in the spectral parameter (the pairs used before were polynomial or rational in the spectral parameter), cf [6, 28]. It is also closely related to some finite-dimensional integrable systems, see [30]. Though the LL equation has been intensively studied, there are still some points that deserve to be cleared up. One of them is related to the fact that the LL equation also admits an L-A pair that is polynomial in the spectral parameter. So there are two different ways of approaching the Landau–Lifshitz equation, the questions of integrability, conservation laws, etc and it is not known if they are equivalent or not. More specifically, the polynomial pair has still not been applied to find exact solutions and it is not known if it will give the same class of solutions as the elliptic pair.

Another point is that when $r_i \to 0$ the LL equation transforms into the Heisenberg ferromagnet equation (HFeq), see (13). The HFeq is gauge equivalent to the nonlinear Schrödinger equation (NLSeq) [6, 10, 35] and is nearly as famous as the NLSeq itself. However, some of the objects related to the LLeq do not go to the analogous objects for the HFeq when we pass to the limit $r_i \to 0$ and the reason for this is not completely understood.

In a sequence of papers [2, 3, 33], we initiated the study of polynomial pencils of Lax pairs leading to hierarchies of soliton equations that include the $O(3)$ chiral system equations and the Landau–Lifshitz equation. Our interest was motivated by what was said in the above, that is, by the problem whether the elliptic L-A pairs and the polynomial L-A pairs are equivalent. Certainly, the first question one can ask is whether the hierarchies of equations obtained via polynomial pencil and via elliptic pencil are equivalent, that is, whether the vector fields corresponding to one of the hierarchies is a finite linear combination of the vector fields corresponding to the equations from the other hierarchy. To this end the study of the corresponding recursion operators is needed. We have found the recursion operators for the chiral fields equations hierarchy (CF hierarchy) related to the polynomial pencil in [33] and now we consider the recursion operators for the polynomial pencil in which the LLeq is included.

The main results of the present paper are the following:

- The recursion operators for the Landau–Lifshitz equation hierarchy related to a polynomial pencil (LLp hierarchy) are calculated—section 5, proposition 6, formula (86).
- It is shown that the dual of the recursion operator for the LLp hierarchy is a Nijenhuis operator and the manifold of potentials has structure of a P–N manifold (section 6, theorem 4).
- It is proved that the LLp hierarchy consists of Hamiltonian vector fields and these fields have commuting flows. This fact results from the existence of the P–N structure but we provide an alternative proof (section 6, theorem 4, corollary 3).

The properties listed in the last two items are characteristic for the integrable systems (soliton equations), [6, 17], in other words, our results prove that the polynomial pencil we investigate has the usual features of the pencils of L-A pairs used in soliton equations theory.

The paper is organized as follows. In section 2 we introduce the hierarchy of the equations related to LLeq when one uses polynomial pencil (to distinguish the elliptic case and the polynomial one we call it the LLp hierarchy). In this section, we discuss some non-canonical algebraic structures, paying a special attention to the case of the algebra $so(4)$, since we shall need later the properties of the new Lie algebra brackets that can be defined on it. In sections 4 and 5 we study the LLp hierarchy and obtain the corresponding recursion operators. Since in their definition appears the inverse of some other operator, we discuss the existence of this inverse. We also compare the recursion operators one gets via elliptic and via polynomial pencil. In the last section (section 6) we give a geometric interpretation of the recursion operators we have obtained. More specifically, we find two compatible Poisson structures such that the LLp hierarchy consists of equations that are Hamiltonian for both these structures (with different Hamiltonians of course). These Poisson structures are closely related to the non-canonical brackets on $so(4)$ we described above. Not surprisingly, we are able to prove that the dual of the recursion operator relates the two Poisson structures, that is, we prove that we have the already typical situation when the integrable hierarchy of equations corresponds to a hierarchy of fundamental fields of a Poisson–Nijenhuis structure on the manifold of potentials.

Now, few words about the terminology. Usually, the soliton equations appear in hierarchies, which have the form:

$$\begin{aligned} \dot{q}(t) &= X_n(q(t)) = P_q(\gamma_n)(q(t)), \qquad q \in \mathcal{M} \\ \gamma_{n+1} &= \Lambda(\gamma_n), \qquad n = 0, 1, 2 \ldots, \end{aligned} \tag{2}$$

where

- $\mathcal{M}$ is an infinite-dimensional manifold of some functions defined on the real line, called the manifold of potentials, that is, the point q is actually a function $q(x)$ and the above equations are partial differential equations.
- γ_n are closed 1-forms on $\mathcal{M}$ (even exact forms, that is, $\gamma_n = -\mathrm{d}H_n$, where H_n are the Hamiltonians of the equations (2), the last being true at least starting from some n).
- $q \mapsto P_q : \mathrm{Hom}(T_q^*(\mathcal{M}), T_q(\mathcal{M}))$ is a Poisson tensor field, see (88).
- X_n are vector fields (dynamic systems on $\mathcal{M}$).
- $q \mapsto \Lambda_q : \mathrm{End}(T_q^*(\mathcal{M}))$ is a field of operators.

If we have the above situation, we shall say that we have a *recursion operator* Λ. Sometimes the hierarchy is described in terms of $N = \Lambda^*$, then it is also called a recursion operator, as a matter of fact, usually one has $P \circ \Lambda = N \circ P$. We say that the hierarchy (2) admits a *bi-Hamiltonian formulation* or simply is bi-Hamiltonian, if it can be written in a Hamiltonian form using another Poisson tensor field Q (different from P) and the Poisson fields P and Q are compatible, that is, if their Schouten bracket $[P, Q]_S$ vanishes, see (89). Usually one has relations of the type:

$$Q(\gamma_{n+1}) + P(\gamma_n) = 0, \qquad n = 0, 1, 2 \ldots, \tag{3}$$

which we call *chain systems*. Chain systems are obtained from the pencils of Lax pairs for the hierarchy (3), but can be obtained also independently. In case the hierarchy is bi-Hamiltonian and one can invert Q, one can construct $\Lambda = Q^{-1} \circ P$ and it can be shown that $N = \Lambda^*$ is *a Nijenhuis tensor* (also called *hereditary operator*) and P, N endow $\mathcal{M}$ with (*Poisson–Nijenhuis structure* (P–N *structure*); see [16, 17] for more profound discussion. Thus the P–N structure implies that N is a hereditary operator and $N^* = \Lambda$ (or N) is a recursion operator. We note also that sometime different authors use the terms recursion operator, hereditary operator, Nijenhuis operator in somewhat different ways.

2. Preliminaries

In order to present the LLp hierarchy, we need some facts and notations. Let $so(4)$ be the algebra of 4×4 skew-symmetric matrices over $\mathbb{C}$. For $\mathbf{u} \in \mathbb{C}^3$ we write

$$\mathbf{u} \to \{\mathbf{u}\}_I = \begin{pmatrix} 0 & u_1 & u_2 & u_3 \\ -u_1 & 0 & u_3 & -u_2 \\ -u_2 & -u_3 & 0 & u_1 \\ -u_3 & u_2 & -u_1 & 0 \end{pmatrix} \tag{4}$$

$$\mathbf{v} \to \{\mathbf{v}\}_{II} = \begin{pmatrix} 0 & v_1 & v_2 & -v_3 \\ -v_1 & 0 & v_3 & v_2 \\ -v_2 & -v_3 & 0 & -v_1 \\ v_3 & -v_2 & v_1 & 0 \end{pmatrix}. \tag{5}$$

Every element $A \in so(4)$ can be written into the form

$$A = \{\mathbf{u}\}_I + \{\mathbf{v}\}_{II}, \tag{6}$$

and this representation corresponds to the well-known splitting of $so(4)$ into the direct sum of two $so(3)$ algebras. In other words, the subalgebras

$$\begin{aligned} \mathfrak{G}_I &= \{\{\mathbf{u}\}_I : \mathbf{u} \in \mathbb{C}^3\} \subset so(4) \\ \mathfrak{G}_{II} &= \{\{\mathbf{u}\}_{II} : \mathbf{u} \in \mathbb{C}^3\} \subset so(4) \end{aligned} \tag{7}$$

are both isomorphic to the algebra $so(3)$ and are ideals in $so(4)$. As a consequence, $[\mathfrak{G}_I, \mathfrak{G}_{II}] = 0$. The polynomial pencil of Lax pairs we are going to study is then written as follows:

$$\begin{aligned} L &\equiv \frac{\partial}{\partial x} - U, \qquad M_n \equiv \frac{\partial}{\partial t} - V_n \\ U(\lambda) &= \frac{1}{2} A(\lambda + J) \end{aligned} \tag{8}$$

$$\begin{aligned} V_n(\lambda) &= \frac{1}{2}(\lambda^n B_0 + \lambda^{n-1} B_1 + \cdots + B_n)(\lambda + J) \\ n &= 0, 1, 2, \ldots, \end{aligned} \tag{9}$$

where

$$A = \{\mathbf{S}\}_I, \qquad B_n = \{\mathbf{b}_n\}_I + \{\mathbf{c}_n\}_{II}, \tag{10}$$

J is the diagonal matrix

$$J = \mathrm{diag}(-j_1 - j_2 + j_3, -j_1 + j_2 - j_3, j_1 - j_2 - j_3, j_1 + j_2 + j_3), \tag{11}$$

and $\mathbf{S}(x) \in \mathbb{R}^3$ is a smooth vector field depending on one spatial variable, taking values on the unit sphere $\mathbb{S}^2 = \{\mathbf{S} : \mathbf{S}^2 = 1\}$ and tending fast enough to some limit value $\mathbf{S}_0$ when $|x| \to \infty$. Of course, the derivatives of $\mathbf{S}(x)$ go to zero when $|x| \to \infty$. Usually $\mathbf{S}_0$ is assumed to be $(0, 0, 1)$, which we shall also assume. We require a condition that ensures the above, but it is stronger, we assume that

$\mathbf{S}(x)$ *is a function taking its values on the sphere* $\mathbb{S}^2$, *such that the components of* $\mathbf{S}(x) - \mathbf{S}_0$ *are Schwartz-type functions on the line. We write all this shortly as*

$$(sw) \lim \mathbf{S}(x) = \mathbf{S}_0 = (0, 0, 1). \tag{12}$$

Two famous infinite-dimensional integrable systems exist on the space $\mathcal{M}_S$ of the functions $\mathbf{S}(x)$, satisfying the above conditions, which attracted considerable attention in the past decades:

(i) The Heisenberg ferromagnet equation (HFeq):

$$\mathbf{S}_t = \mathbf{S} \times \mathbf{S}_{xx}. \tag{13}$$

(ii) The Landau–Lifshitz equation (LLeq) we introduced in (1).

Lax pairs for both HFeq and LLeq can be obtained in the form (8), so we look now more closely into this hierarchy of evolution equations. Let us denote the set of the matrices of the type $A = \{\mathbf{S}\}_I$, with potential $\mathbf{S}(x)$ belonging to $\mathcal{M}_S$ by $\mathcal{M}_S^I$. In other words, $\mathcal{M}_S^I \subset so(4)$ is the manifold of potentials. In [3] we have found that the set of the nonlinear evolution equations, corresponding to the above Lax pairs, have the form:

$$A_t = (B_n)_x - \tfrac{1}{2}(AJB_n - B_nJA), \tag{14}$$

where the matrix functions B_n satisfy the following infinite system of equations which we shall call the LLp chain system ('p' to denote that it is obtained through polynomial pencil):

$$\begin{aligned} &[A, B_0] = 0 \\ &[A, B_{n+1}] = 2(B_n)_x - (AJB_n - B_nJA), \qquad n = 0, 1, \ldots. \end{aligned} \tag{15}$$

The equations (14) can be written also into the equivalent form

$$A_t = \tfrac{1}{2}[A, B_{n+1}]. \tag{16}$$

The hierarchy (14) or (16) is the Landau–Lifshitz hierarchy of evolution equations, or simply an LL hierarchy, but as we need to distinguish hierarchies obtained through different pencils we call it an LLp hierarchy. Of course, the above relations can be written also in terms of the vector fields $\mathbf{S}$, $\mathbf{b}_n$ and $\mathbf{c}_n$ and take the form:

- The LLp hierarchy of evolution equations:

$$\mathbf{S}_t = (\mathbf{b}_n)_x - \mathbf{S} \times K(\mathbf{c}_n), \qquad n = 0, 1, 2, \ldots, \tag{17}$$

or

$$\mathbf{S}_t = -\mathbf{S} \times \mathbf{b}_{n+1}, \qquad n = 0, 1, 2, \ldots. \tag{18}$$

- The LLp chain system

$$\begin{aligned} &\mathbf{S} \times \mathbf{b}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times K(\mathbf{c}_n) \\ &(\mathbf{c}_n)_x = -K(\mathbf{S} \times \mathbf{b}_n) + K(\mathbf{S}) \times \mathbf{c}_n \\ &n = 0, 1, \ldots. \end{aligned} \tag{19}$$

In the above expressions K is the diagonal matrix

$$K = \mathrm{diag}(j_1, j_2, j_3), \tag{20}$$

where j_i are the same as in the definition of J and $(K(\mathbf{a}))_i \equiv j_i a_i$. Below, when same relations are expressed through 4×4 matrices in $so(4)$, we shall say that we have matrix representation of the hierarchies, equations etc; when expressed in terms of vectors, we shall say that we have vector representation.

It can be shown that the LLp chain system possesses a solution at least if we begin with $\mathbf{b}_0 = \mathbf{S}$, $\mathbf{c}_0 = 0$. We discuss this solution later, now we just note that with this choice the second nontrivial equation of the hierarchy is the Landau–Lifshitz equation

$$\mathbf{S}_t = (\mathbf{b}_1)_x - \mathbf{S} \times \mathbf{c}_1 = \mathbf{S} \times \mathbf{S}_{xx} - \mathbf{S} \times R(\mathbf{S}), \tag{21}$$

where $R = -K^2$, that is,

$$R = \mathrm{diag}(r_1, r_2, r_3), \qquad r_i = -j_i^2, \qquad i = 1, 2, 3. \tag{22}$$

This means that in order to obtain the LL equation the entries of K must be purely imaginary. The equations (14) and (15) reveal the importance of a special Lie algebra structure, which we shall introduce below.

3. The linear pencil of Lie brackets over $so(4)$ and its properties

3.1. The linear pencil of Lie brackets over so(n)

Let $so(n)$ $(n > 2)$ be the Lie algebra of skew-symmetric $n \times n$ matrices (the considerations below are the same both for $\mathbb{R}$ and $\mathbb{C}$, so we do not specify the field of the numbers). Let J be some fixed symmetric matrix. We define then the following bi-linear skew-symmetric map:

$$\begin{aligned} &[\,,\,]_J : so(n) \times so(n) \to so(n) \\ &[X, Y]_J = XJY - YJX. \end{aligned} \tag{23}$$

It is easy to check that $[X, Y]_J$ is a Lie bracket. The vector space $so(n)$ endowed with the bracket $[\,,\,]_J$ is a Lie algebra, which we shall denote by $so(n)_J$. But since the choice $J = \mathbf{1}$ leads to the usual Lie algebra structure, in this case we write just $so(n)$. As a matter of fact, we have here the so-called linear pencil of Lie brackets (also called linear bundle of Lie brackets) related to the algebra $so(n)$, because for each symmetric J we have a Lie bracket. We remind some facts about these structures; for more detail, see [34]. Since $[\,,\,]_{J_1} + [\,,\,]_{J_2} = [\,,\,]_{J_1+J_2}$, the new brackets define compatible Poisson tensors on the dual $so^*(n)$, a fact that is crucial for the construction we are going to present. Also, $(X, Y) \mapsto [X, Y]_J$ is a 2-cocycle for the adjoint representation of $so(n)$ (with its usual structure) and this cocycle is a ad-coboundary, since

$$[X, Y]_J = \mathrm{ad}_X\alpha(Y) - \mathrm{ad}_Y\alpha(X) - \alpha([X, Y]), \tag{24}$$

where $\alpha \in \mathrm{End}(so(n))$ is given by

$$\alpha(X) = \tfrac{1}{2}(XJ + JX), \tag{25}$$

and, as usual, $\mathrm{ad}_X(Y) \equiv [X, Y]$. In case the algebra is $so(4)$, and J is as in (11), the cocycle α has the following properties, which we shall use later.

Proposition 1. *Let J be the diagonal matrix, introduced in (11). Then the map α, defined in (25) interchanges the two $so(3)$ subalgebras of $so(4)$. More precisely,*

$$\alpha(\{\mathbf{u}\}_I) = -\{K\mathbf{u}\}_{II}, \qquad \alpha(\{\mathbf{u}\}_{II}) = -\{K\mathbf{u}\}_I, \tag{26}$$

where $K = \mathrm{diag}(j_1, j_2, j_3)$.

3.2. The algebras $so(4)_J$, invariant bi-linear form

As is well known, non-degenerate invariant symmetric bi-linear forms over Lie algebras play an important role in the construction of the so-called Poisson–Lie brackets. Provided such a form is given we can identify the algebra $\mathfrak{G}$ with its dual space $\mathfrak{G}^*$ and the adjoint representation $X \mapsto \mathrm{ad}_X$ with the coadjoint representation $X \mapsto -\mathrm{ad}_X^*$. Among the algebras $so(n)$ the algebra $so(4)$ is quite exceptional. The point is that on it one can define an inner product, which is invariant simultaneously under all the Lie algebra structures in the pencil introduced above, see [34]. Since our Lax pairs lie in $so(4)$ we shall concentrate on this algebra.

Let J be some fixed symmetric 4×4 matrix and α be the 1-cocycle defined by it. The algebra $so(4)$ is semisimple but not a simple one. A finite-dimensional algebra $\mathfrak{G}$ (over $\mathbb{R}$ or $\mathbb{C}$) is semisimple if its Killing form $B(X, Y)$ is non-degenerate. If the algebra $\mathfrak{G}$ is simple each invariant symmetric bi-linear form is proportional to B. For example, if $\mathfrak{G} = so(n), n > 4$, the form $\mathrm{tr}(XY)$ is symmetric, non-degenerate and invariant. It is well known that $B(X, Y)$ is proportional to $\mathrm{tr}(XY)$. For the algebra $so(4)$ the Killing form is also proportional to the trace form, but there are also another symmetric non-degenerate invariant bi-linear forms which are not proportional to the Killing form. In order to introduce such a form on $\mathfrak{G} = so(4)$ let us define the linear map $T : so(4) \to so(4)$

$$(T(X))_{ij} = \tfrac{1}{2}\epsilon_{ijks} X_{ks} \qquad 1 \leqslant i, j, k, s \leqslant 4. \tag{27}$$

In the above expression ϵ_{ijks} is the skew-symmetric Levi–Cevita symbol and the rule about the summation over repeating indices is assumed. One proves directly some useful properties of B, T and α.

Proposition 2. *The linear map T has the properties:*

(i) T *is involutive:*

$$T^2 = id_{so(4)}. \tag{28}$$

(ii) T *is symmetric with respect to the Killing form:*

$$B(T(X), Y) = B(T(Y), X). \tag{29}$$

(iii) $[T(X), T(Y)] = [X, Y]$ *and the two* $so(3)$ *subalgebras of* $so(4)$ *are invariant under the action of* T*:*

$$T(\{\mathbf{u}\}_I) = \{\mathbf{u}\}_I, \qquad T(\{\mathbf{u}\}_{II}) = -\{\mathbf{u}\}_{II}. \tag{30}$$

(iv) For arbitrary symmetric matrix J*, the form*

$$(X, Y, Z) \mapsto B(XJY - YJX, T(Z)) = B\big(ad^J_X(Y), T(Z)\big) \tag{31}$$

is 3-cocycle for the trivial representation of $so(4)$*, in particular, the form* $B([X, Y], T(Z))$ *is 3-cocycle.*

(v) The linear map α *is symmetric with respect to the Killing form:*

$$B(\alpha(X), Y) = B(X, \alpha(Y)); \qquad X, Y \in so(4). \tag{32}$$

(vi) $T \circ \alpha = -\alpha \circ T$.

With the help of T we define another symmetric bi-linear form

$$B_T(X, Y) \equiv B(X, T(Y)), \qquad X, Y \in so(4). \tag{33}$$

It can be proved that

Proposition 3. B_T *is a symmetric non-degenerate bi-linear form, invariant with respect to the adjoint action of* $so(4)_J$.

Remark 1. Note that there is no contradiction here because B_T *is not* the Killing form of $so(4)$. As a matter of fact, if B_I and B_{II} are the Killing forms of the two $so(3)$-subalgebras of $so(4)$, each extended as zero on the other one, then $B_T = B_I - B_{II}$, while the Killing form B equals $B_I + B_{II}$.

From the properties of T and α, listed in proposition 2, we also get

Proposition 4. *The linear map* α *introduced in (25) is skew-symmetric with respect to the form* B_T*, that is,*

$$B_T(\alpha(X), Y) + B_T(X, \alpha(Y)) = 0; \qquad X, Y \in so(4). \tag{34}$$

Corollary 1. *The form* $(X, Y) \mapsto \Omega_J(X, Y) = B_T(\alpha(X), Y)$ *is skew-symmetric. With respect to it the ideals* $\mathfrak{G}_I$ *and* $\mathfrak{G}_{II}$ *and the 'graphs',*

$$\begin{aligned} \Gamma_I &= \{A + \alpha(A);\ A \in \mathfrak{G}_I\} \\ \Gamma_{II} &= \{A + \alpha(A);\ A \in \mathfrak{G}_{II}\}, \end{aligned} \tag{35}$$

are isotropic subspaces. In particular, if J is as in (11) and if $j_1 j_2 j_3 \neq 0$, the form $\Omega_J(X, Y)$ is a symplectic form and $\mathfrak{G}_I, \mathfrak{G}_{II}, \Gamma_I, \Gamma_{II}$ are Lagrangian subspaces.

We note also that for arbitrary $\mathbf{a}, \mathbf{b} \in \mathbb{C}^3$

$$\begin{aligned} &B(\{\mathbf{a}\}_I, \{\mathbf{b}\}_I) = B_T(\{\mathbf{a}\}_I, \{\mathbf{b}\}_I) = -8\langle \mathbf{a}, \mathbf{b}\rangle \\ &B(\{\mathbf{a}\}_{II}, \{\mathbf{b}\}_{II}) = -B_T(\{\mathbf{a}\}_{II}, \{\mathbf{b}\}_{II}) = -8\langle \mathbf{a}, \mathbf{b}\rangle \\ &B(\{\mathbf{a}\}_I, \{\mathbf{b}\}_{II}) = B_T(\{\mathbf{a}\}_I, \{\mathbf{b}\}_{II}) = 0, \end{aligned} \tag{36}$$

where $\langle \mathbf{a}, \mathbf{b}\rangle = \sum_{i=1}^{3} a_i b_i$.

4. The LL hierarchy, obtained via polynomial pencil

Let us consider now the LLp chain system (15). Using the properties of the cocycle α, we first cast it in the following form:

$$\begin{aligned} &[A, F_0] = 0 \\ &\tfrac{1}{2}[A, F_{n+1}] = (F_n)_x - \tfrac{1}{2}[A, \alpha(G_n)] \\ &(G_n)_x - \tfrac{1}{2}[\alpha(A), G_n] + \tfrac{1}{2}\alpha([A, F_n]) = 0 \\ &n = 0, 1, 2, \ldots \end{aligned} \tag{37}$$

$$A = \{\mathbf{S}\}_I, \qquad F_n = \{\mathbf{b}_n\}_I, \qquad G_n = \{\mathbf{c}_n\}_{II}, \tag{38}$$

or, equivalently, [3]:

$$\begin{aligned} &\mathbf{S} \times \mathbf{b}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times K(\mathbf{c}_n) \\ &(\mathbf{c}_n)_x - K(\mathbf{S}) \times \mathbf{c}_n = -K(\mathbf{S} \times \mathbf{b}_n) \\ &n = 0, 1, 2, \ldots . \end{aligned} \tag{39}$$

Here, as in the rest of the text, $K = \mathrm{diag}(j_1, j_2, j_3)$. It is known, see [3], that the LLp chain system has the following solution (obtained recursively):

$$\begin{aligned} &\mathbf{b}_0 = \mathbf{S}, \qquad \mathbf{c}_0 = 0 \\ &\mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x, \qquad \mathbf{c}_1 = K(\mathbf{S}) \\ &\mathbf{b}_{n+1} = \mathbf{b}^S_{n+1} + \mathbf{S} \int_{\pm\infty}^{x} \langle \mathbf{b}^S_{n+1}, \mathbf{S}_x \rangle \, \mathrm{d}x \\ &\mathbf{b}^S_{n+1} = \Lambda_\pm(\mathbf{b}^S_n) + (K(\mathbf{c}_n))^S \end{aligned} \tag{40}$$

$$\begin{aligned} &\mathbf{c}_n = \sum_{q=1}^{n} (-1)^{q-1} K^{(q)}(\mathbf{b}_{n-q}) \\ &n = 1, 2, \ldots . \end{aligned} \tag{41}$$

Below we explain the meaning of the notations and quantities that appear in the solution. First, the superscript S means that we take the projection to the space orthogonal to S, that is,

$$\mathbf{a}^S \equiv \mathbf{a} - \langle \mathbf{a}, \mathbf{S}\rangle \mathbf{S} = -\mathbf{S} \times (\mathbf{S} \times \mathbf{a}). \tag{42}$$

Next, the sequence of diagonal matrices $K^{(n)}$, $n = 1, 2, \ldots$ is constructed recursively, starting from $K^{(1)} \equiv K$. More specifically, the matrices $K^{(n)}$ are obtained from the requirement that for arbitrary $\mathbf{a}$ and $\mathbf{b}$ they satisfy the following equations:

$$
\begin{aligned}
&K^{(1)} \equiv K \\
&K^{(1)}(\mathbf{a}) \times K^{(1)}(\mathbf{b}) = K^{(2)}(\mathbf{a} \times \mathbf{b}) \\
&K^{(1)}(\mathbf{a} \times K^{(1)}K^{(1)}(\mathbf{b})) + K^{(1)}(\mathbf{a}) \times K^{(2)}(\mathbf{b}) = K^{(3)}(\mathbf{a} \times \mathbf{b}) \\
&K^{(1)}(\mathbf{a} \times K^{(1)}K^{(2)}(\mathbf{b})) + K^{(2)}(\mathbf{a} \times K^{(1)}K^{(1)}(\mathbf{b})) + K^{(1)}(\mathbf{a}) \times K^{(3)}(\mathbf{b}) = K^{(4)}(\mathbf{a} \times \mathbf{b}) \\
&\cdots\cdots\cdots\cdots\cdots\cdots \\
&\sum_{i=1}^{n-2} K^{(i)}(\mathbf{a} \times K^{(1)}K^{(n-i-1)}(\mathbf{b})) + K^{(1)}(\mathbf{a}) \times K^{(n-1)}(\mathbf{b}) = K^{(n)}(\mathbf{a} \times \mathbf{b}) \\
&n = 3, 4, \ldots.
\end{aligned}
\tag{43}
$$

The family of diagonal matrices $K^{(n)}$ is well defined, see [3], and the entries $K_i^{(n)}$, $i = 1, 2, 3$ of $K^{(n)}$ are homogeneous polynomials of degree n in the variables j_1, j_2, j_3. For example, the first members of the family $K^{(n)}$ are

$$
\begin{aligned}
&K_1^{(1)} = j_1, \quad K_1^{(2)} = j_2 j_3, \quad K_1^{(3)} = j_1\left(j_2^2 + j_3^2\right), \quad K_1^{(4)} = j_2 j_3\left(2j_1^2 + j_2^2 + j_3^2\right) \\
&K_2^{(1)} = j_2, \quad K_2^{(2)} = j_1 j_3, \quad K_2^{(3)} = j_2\left(j_1^2 + j_3^2\right), \quad K_2^{(4)} = j_1 j_3\left(j_1^2 + 2j_2^2 + j_3^2\right) \\
&K_3^{(1)} = j_3, \quad K_3^{(2)} = j_1 j_2, \quad K_3^{(3)} = j_3\left(j_1^2 + j_2^2\right), \quad K_3^{(4)} = j_1 j_2\left(j_1^2 + j_2^2 + 2j_3^2\right).
\end{aligned}
\tag{44}
$$

Note that $KK^{(2)} = j_1 j_2 j_3 \mathbf{1}$ and that if $K = \mathbf{1}$, all the matrices $K^{(n)}$ are proportional to $\mathbf{1}$. If we already have the matrices $K^{(n)}$, it can be shown that $\mathbf{c}_n$ is expressed as shown in (41) through $\mathbf{b}_s$, $0 \leqslant s \leqslant n-1$. We shall assume that the components of $\mathbf{b}_n$ and $\mathbf{c}_n$ are polynomials in the components of $\mathbf{S}, \mathbf{S}_x, \mathbf{S}_{xx}, \ldots$. Similar facts are typical in the theory of the soliton equations. However, one proves them later, after more careful examination of the properties of the corresponding recursion operators, see for example [9] for the case of the Zakharov–Shabat system and its gauge-equivalent, or follow from some other considerations, usually related to the auxiliary spectral problems. If we take such dependence on $\mathbf{S}(x)$ and its derivatives as granted, then the matrices B_n have the same behaviour at $+\infty$ and $-\infty$:

$$
\lim_{x \to -\infty} B_n(x) = \lim_{x \to +\infty} B_n(x).
\tag{45}
$$

Finally, the operators $\Lambda_\pm$ are given by the formula

$$
\Lambda_\pm(\mathbf{X}(x)) \equiv \mathbf{S} \times \frac{\partial}{\partial x}\mathbf{X}(x) + (\mathbf{S} \times \mathbf{S}_x) \int_{\pm\infty}^{x} \langle \mathbf{X}(x), \mathbf{S}_x \rangle \, \mathrm{d}x,
\tag{46}
$$

and are the recursion operators for the FH chain system, familiar from the study of the Heisenberg ferromagnet equation (see for example [6, 10] or [1]). The FH chain system has the form

$$
\begin{aligned}
&\mathbf{S} \times \mathbf{b}_0 = 0 \\
&\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x \\
&n = 0, 1, \ldots
\end{aligned}
\tag{47}
$$

and can be obtained as a limit case of the LLp chain system putting $K = 0$, $\mathbf{c}_n = 0$. In other words, the solution of (47) is

$$\begin{aligned}
&\mathbf{b}_0 = \mathbf{S}, \qquad \mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x \\
&\mathbf{b}^S_{n+1} = \Lambda_\pm\big(\mathbf{b}^S_n\big) \\
&\mathbf{b}_{n+1} = \mathbf{b}^S_{n+1} + \mathbf{S}\int_{\pm\infty}^{x}\big\langle \mathbf{b}^S_{n+1}, \mathbf{S}_x\big\rangle\,\mathrm{d}x \\
&n = 1, 2, \ldots.
\end{aligned} \tag{48}$$

Coming back to the LLp chain system, the first few members of the solution are

$$\begin{aligned}
&\mathbf{c}_1 = K(\mathbf{S}) \\
&\mathbf{b}_2 = \mathbf{S}_{xx} - K^2(\mathbf{S}) + h_{LL}\mathbf{S} \\
&\mathbf{c}_2 = K(\mathbf{S} \times \mathbf{S}_x) - K^{(2)}(\mathbf{S}) \\
&\mathbf{b}_3 = -\mathbf{S} \times \mathbf{S}_{xxx} + \mathbf{S} \times K^2(\mathbf{S}_x) + K^2(\mathbf{S} \times \mathbf{S}_x) + h_{LL}\mathbf{S} \times \mathbf{S}_x - \langle K^2(\mathbf{S} \times \mathbf{S}_x), \mathbf{S}\rangle\mathbf{S} \\
&\mathbf{c}_3 = K(\mathbf{b}_2) - K^{(2)}(\mathbf{b}_1) + K^{(3)}(\mathbf{b}_0) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots
\end{aligned} \tag{49}$$

where

$$h_{LL} \equiv \tfrac{1}{2}\langle \mathbf{S}_x, \mathbf{S}_x\rangle - \big(\tfrac{1}{2}\langle K^2(\mathbf{S}), \mathbf{S}\rangle - \tfrac{1}{2}\langle K^2(\mathbf{S}_0), \mathbf{S}_0\rangle\big). \tag{50}$$

The notation h_{LL} is chosen because h_{LL} is the Hamiltonian density for the LL equation with respect to the so-called first Hamiltonian structure on $\mathcal{M}_S$ (see the first equation in (110)). This means that the Hamiltonian function is equal to

$$H_{LL} = \int_{-\infty}^{+\infty} h_{LL}(\mathbf{S}(x), \mathbf{S}_x(x))\,\mathrm{d}x. \tag{51}$$

It is interesting to note the HF equation can be embedded in the LLp hierarchy not only if we set $K = 0$, but also if we set $K = \mathbf{1}$. Then the HF equation is the second nontrivial equation in the hierarchy. The corresponding chain system, to which we refer as the HFI chain system, is

$$\begin{aligned}
&\mathbf{S} \times \mathbf{b}_0 = 0 \\
&\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times \mathbf{c}_n \\
&(\mathbf{c}_n)_x - \mathbf{S} \times \mathbf{c}_n = -\mathbf{S} \times \mathbf{b}_n \\
&n = 0, 1, \ldots,
\end{aligned} \tag{52}$$

and as far as we know, never it has been considered in relation to the HF equation. One can solve the HFI chain system but, as can be expected, the solution does not yield new equations. Indeed, if the nth equation of HFI is $\mathbf{S}_t = Y_n(\mathbf{S}, \mathbf{S}_x, \ldots)$ and the nth equation of the HF hierarchy is $\mathbf{S}_t = X_n(\mathbf{S}, \mathbf{S}_x, \ldots)$, then Y_n is a linear combination with coefficients that do not depend on x (but depend on n) of $X_1, X_2, \ldots, X_n$. This of course can be seen directly from the pencil of the Lax pairs; however, for reasons that will become clear later, we do it in terms of the chain system. When $K = \mathbf{1}$, all the matrices $K^{(s)} = k_s\mathbf{1}$, where k_s are positive integers, obtained recursively as follows:

$$\begin{aligned}
&k_1 = k_2 = 1 \\
&k_{n+2} = k_1k_n + k_2k_{n-1} + \cdots + k_{n-1}k_2 + k_nk_1 + k_{n+1} \\
&n = 1, 2, \ldots.
\end{aligned} \tag{53}$$

For example, $k_3 = 2, k_4 = 4, k_5 = 9, k_6 = 21$. Then for $n = 1, 2, \ldots$ we get

$$\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S} \times \sum_{s=1}^{n} (-1)^{s-1} k_s \mathbf{b}_{n-s}, \tag{54}$$

and therefore,

$$\begin{aligned} \mathbf{b}_{n+1}^S &= \Lambda_\pm\big(\mathbf{b}_n^S\big) + \sum_{s=1}^{n} (-1)^{s-1} k_s \mathbf{b}_{n-s}^S \\ \mathbf{b}_{n+1} &= \mathbf{b}_{n+1}^S + \mathbf{S} \int_{\pm\infty}^{x} \big\langle \mathbf{b}_{n+1}^S, \mathbf{S}_x \big\rangle \, \mathrm{d}x. \end{aligned} \tag{55}$$

Since as before $\mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x$, the elements of the new hierarchy are linear combinations with constant coefficients of the elements of the hierarchy HF. We can also try to find a recursion operator here. From the HFI chain system we get

$$\mathbf{b}_{n+1}^S = \Lambda_\pm\big(\mathbf{b}_n^S\big) + \mathbf{c}_n^S. \tag{56}$$

Unfortunately, when we want to express $\mathbf{c}_n^S$, the formula is more complicated:

$$\mathbf{b}_n^S = \mathbf{c}_n^S + \Lambda_\pm\big(\mathbf{c}_n^S\big) + (-1)^{n-1} k_n \mathbf{S} \times \mathbf{S}_x. \tag{57}$$

We can write formally

$$\mathbf{c}_n^S = (\mathrm{id} + \Lambda_\pm)^{-1} \big[\mathbf{b}_n^S + (-1)^n k_n \mathbf{S} \times \mathbf{S}_x\big], \tag{58}$$

but the above relation cannot be used to calculate recursion operators, since it involves the numbers k_n, which are found using another recursion process. However, there is other way of approaching the problem, that works for arbitrary K, which we are going to present in the following section.

Remark 2. Of course, additional efforts are needed here in order to give meaning to the operator $(\mathrm{id} + \Lambda_\pm)^{-1}$, since as is known, [10], the continuous spectrum of $\Lambda_\pm$ fills up the real line. The considerations in [10] are done for the algebra $sl(2, \mathbb{C})$, but since $su(2) \sim \mathbb{R}^3$ (as algebras), then also $\mathbb{C}^3 \sim sl(2, \mathbb{C})$ and it is easy to 'translate' all the results and formulae to the case when we are working in $\mathbb{C}^3$ or $\mathbb{R}^3$.

5. The recursion operator for the LLp hierarchy

Let us first compare the LLp chain system with the chain system obtained in [1] for elliptic pencil of Lax pairs. We shall call it the LLe chain system. It is given by

$$\begin{aligned} &\mathbf{S} \times \mathbf{a}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_n = (\mathbf{a}_n)_x \\ &\mathbf{S} \times \mathbf{a}_{n+1} = \tfrac{1}{4}[(\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n] \\ &n = 0, 1, 2, \ldots. \end{aligned} \tag{59}$$

Here C is the diagonal matrix,

$$C = \mathrm{diag}(r_1 - r_3, r_2 - r_3, 0) = R - r_3\mathbf{1} \tag{60}$$

and the hierarchy is calculated starting from $\mathbf{a}_0 = -\mathbf{S}$. Then $\mathbf{b}_0 = \mathbf{S} \times \mathbf{S}_x$, and the LLe hierarchy of evolution equations is given by

$$\begin{aligned} &\mathbf{S}_t = (\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n \\ &n = 0, 1, 2, \ldots. \end{aligned} \tag{61}$$

The first equation in the hierarchy is the LL equation. The recursion operator, obtained in [1], and since then considered in the literature as the recursion operator for the LL equation, has the form

$$\Phi^{LL}_{\pm}(\mathbf{h}) = \tfrac{1}{4}(\Lambda_{\pm})^2\mathbf{h} - \tfrac{1}{4}\mathbf{S} \\ \times \left\{C(\mathbf{S}) \times \mathbf{h} + (\mathbf{S}_x)\partial_x^{-1}(\langle \mathbf{S}, C(\mathbf{S}) \times \mathbf{h}\rangle) + C(\mathbf{S}) \times \mathbf{S}\partial_x^{-1}(\langle \mathbf{S}, (\mathbf{h})_x\rangle)\right\}, \qquad (62)$$

where it is assumed that $\mathbf{h}^S = \mathbf{h}$. According to the explanations of the authors, ∂_x^{-1} is the inverse of the x-derivative operator ∂_x and $\partial_x^{-1}(0)$ is understood as constant. (See the discussion after $\mathcal{A}_-$ has been introduced in (77).) Then

$$\mathbf{c}_{n+1} = \Phi^{LL}_{\pm}(\mathbf{c}_n), \qquad n = 0, 1, 2, \ldots, \qquad (63)$$

where

$$\mathbf{c}_n \equiv -\mathbf{S} \times [(\mathbf{b}_n)_x + C(\mathbf{S}) \times \mathbf{a}_n]. \qquad (64)$$

In [1] it has been shown by direct calculations that the adjoint operator $\left(\Phi^{LL}_{\pm}\right)^*$ is a Nijenhuis tensor field relating two Poisson structures.

Let us come back to LLp. Though the formulae (40) permit to calculate the hierarchy, it is good to have a recursion operator. This operator must relate $\mathbf{b}^S_{n+1}$ and $\mathbf{b}^S_n$ as does the operator $\Lambda_\pm$ for the HF chain system. We start with the observation that in the equations (39) appears the operator $\mathcal{A}$:

$$\mathcal{A}(\mathbf{h}) \equiv \frac{\mathrm{d}\mathbf{h}}{\mathrm{d}x} - K(\mathbf{S}) \times \mathbf{h}, \qquad (65)$$

which relates $\mathbf{c}_n$ and $\mathbf{b}_n$. If it is invertible, one can eliminate $\mathbf{c}_n$ and arrive at the desired result, just as has been done for the HFI system. For this reason let us consider $\mathcal{A}$ more closely. The natural domain D_n of $\mathcal{A}$ consists of the differentiable functions $\mathbf{h} : \mathbb{R} \mapsto \mathbb{C}^3$. However, since we expect to use $\mathcal{A}$ in the LLp chain system, we restrict D_n to the set D of the functions $\mathbf{h}(x)$, such that $(sw)\lim \mathbf{h}(x) = \left(0, 0, h_3^0\right)$. (Here h_3^0 is some constant). In order to perform our constructions we need additional assumptions on $\mathbf{S}(x)$. We shall assume not only that $(sw)\lim \mathbf{S}(x) = (0, 0, 1)$, but also that

$$|S_{1,2}(x)| \leqslant H \exp(-(|\mathrm{Im}(j_3)| + \epsilon)|x|), \qquad (66)$$

where H, ϵ are some positive constants. (Less restrictive conditions can be introduced, but since we want only to justify our constructions and not to develop the spectral theory of $\mathcal{A}$, we shall use these ones.)

Proposition 5. *If the potential function* $\mathbf{S}(x)$ *satisfies the above conditions, then for any fixed constant* h_3^0 *there exists unique* $\mathbf{h}(x) \in D_n$*, such that* $\mathcal{A}\mathbf{h} = 0$*,* $\lim_{x\to-\infty} \mathbf{h}(x) = \left(0, 0, h_3^0\right)$*.*

Proof. Let $\mathbf{h} \in D_n$ be such that

$$\begin{aligned} &\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}x} - K(\mathbf{S}) \times \mathbf{h} = 0 \\ &\lim_{x\to-\infty} \mathbf{h}(x) = \left(0, 0, h_3^0\right). \end{aligned} \qquad (67)$$

We reformulate our problem in order to put it in a more convenient form. From the algebra $\mathbb{C}^3$ (with respect to the cross-product), we pass to the isomorphic algebra $sl(2)$, that is instead of $\mathbf{h}(x)$, $\mathbf{S}(x)$ we introduce the matrix functions $h(x)$, $S(x)$ etc defined as

$$h(x) = \sum_{s=1}^{3} h_s(x)\sigma_s$$
$$S(x) = \sum_{s=1}^{3} S_s(x)\sigma_s, \qquad S_0 = \sigma_3 \tag{68}$$
$$K(S) = \sum_{s=1}^{3} j_s S_s(x)\sigma_s,$$

where σ_s, $1 \leqslant s \leqslant 3$ are the Pauli matrices. The inner product $\langle \mathbf{g}, \mathbf{h}\rangle = \sum_{s=1}^{3} h_s g_s$ of two vectors $\langle \mathbf{g}, \mathbf{h}\rangle$ can be written as $2B(h, g)$ where $B(h, g)$ is the Killing form of $sl(2)$. Indeed, as is known, for $sl(2)$ holds $B(h, g) = \frac{1}{4}\mathrm{tr}(hg)$ and so $B(\sigma_s, \sigma_l) = \frac{1}{2}\delta_{sl}$. We denote by the same letter the operator that corresponds to $\mathcal{A}$ under the isomorphism $\mathbb{C}^3 \sim sl(2)$, that is, when $\mathcal{A}$ acts on $sl(2)$-valued functions, it takes the form

$$\mathcal{A}(h) = \frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{i}}{2}[K(S), h]. \tag{69}$$

Then the equation (67) reads

$$\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{i}}{2}[K(S), h] = 0. \tag{70}$$

Next we consider the following auxiliary linear problem, closely related to (70):

$$\begin{aligned} &\frac{\mathrm{d}\psi}{\mathrm{d}x} + \frac{\mathrm{i}}{2}K(S)\psi = 0 \\ &\lim_{x\to-\infty} \exp\left(-\mathrm{i}x\frac{j_3}{2}\sigma_3\right)\psi(x) = \mathbf{1}. \end{aligned} \tag{71}$$

If we put $\tilde{\psi}(x) = \exp\left(-\mathrm{i}x\frac{j_3}{2}\sigma_3\right)\psi(x)$, then as easily checked, (71) is equivalent to the following integral equation for $\tilde{\psi}(x)$:

$$\tilde{\psi}(x) = \mathbf{1} - \frac{\mathrm{i}}{2}\int_{-\infty}^{x} \mathrm{e}^{(-\mathrm{i}y\frac{j_3}{2}\sigma_3)}(K(S) - j_3\sigma_3)\,\mathrm{e}^{(\mathrm{i}y\frac{j_3}{2}\sigma_3)}\tilde{\psi}(y)\,\mathrm{d}y, \tag{72}$$

or

$$\tilde{\psi}(x) = \mathbf{1} - \frac{\mathrm{i}}{2}\int_{-\infty}^{x} \begin{pmatrix} j_3(S_3(y)-1) & (j_1S_1(y) - ij_2S_2(y))\,\mathrm{e}^{\mathrm{i}yj_3} \\ (j_1S_1(y) - ij_2S_2(y))\,\mathrm{e}^{-\mathrm{i}yj_3} & -j_3(S_3(y)-1) \end{pmatrix} \tilde{\psi}(y)\,\mathrm{d}y. \tag{73}$$

The assumption (66) ensures that the above integral equation is of Volterra type, since its kernel (with respect to the matrix norm) is bounded by $M\exp(-\epsilon|x|)$, M = constant. Therefore, (66) has a unique solution and it is an invertible matrix. (For the properties of such integral equations see [6], the above linear problem is similar to the auxiliary problem for the Heisenberg ferromagnet equation.) As a consequence, (71) has a unique solution and it is an invertible matrix too. Suppose ψ is a solution of (71). Then the following properties are easily verified:

- If C is a constant 2×2 matrix, then $\psi C\psi^{-1}(x)$ satisfies the equation (70).
- Let $B(X, Y)$ be the Killing form of the algebra $sl(2)$. Suppose $h_1(x), h_2(x)$ are arbitrary functions with values in $sl(2)$ satisfying (70). Then the derivative of $r(x) \equiv B(h_1(x), h_2(x))$ vanishes, that is, $r(x)$ = constanat.

From the above easily follows, that if $h(x)$ is a function with values in $sl(2)$ satisfying (70), it is necessarily of the form:

$$h(x) = \sum_{s=1}^{3} a_s \psi \sigma_s \psi^{-1}(x), \qquad a_s = \text{constant}. \tag{74}$$

Since the asymptotic when $x \to -\infty$ of the right-hand side is

$$(a_1 \cos(j_3 x) + a_2 \sin(j_3 x))\sigma_1 + (-a_1 \sin(j_3 x) + a_2 \cos(j_3 x))\sigma_2 + a_3\sigma_3, \tag{75}$$

the function $h(x)$ can tend to h_3^0 only if $a_s = 0$ $s = 1, 2$ and $a_3 = h_3^0$. The proposition is proved. □

Corollary 2. *If the limit* $\lim_{|x|\to\infty} f(x)$ *is known, the function* $f(x) \in D$ *can be recovered uniquely from its image* $g = \mathcal{A}(f)$. *In particular, restricted to the vector space of Schwartz-type functions, the operator* $\mathcal{A}$ *has trivial kernel and is invertible.*

As to the calculation the inverse operator $\mathcal{A}^{-1}$, it can be done using the function $\psi(x)$. Indeed, consider (70) with a right side, that is,

$$\frac{\mathrm{d}h}{\mathrm{d}x} + \frac{\mathrm{i}}{2}[K(S), h] = g(x), \tag{76}$$

where $g(x)$ is some Schwartz-type function. Then, since $B(\sigma_s, \sigma_l) = -\frac{1}{4}\delta_{sl}$, one can check that we have the following solution of our problem (provided that the integral converges and we can differentiate it):

$$\begin{aligned} h(x) &= \mathcal{R}_{(-)} g(x) \\ &= a\psi\sigma_3\psi^{-1}(x) + \frac{1}{2}\sum_{s=1}^{3} \psi\sigma_s\psi^{-1}(x) \int_{-\infty}^{x} B(g(y), \psi\sigma_s\psi^{-1}(y))\,\mathrm{d}y. \end{aligned} \tag{77}$$

As it should be, the solution depends on one arbitrary parameter a. If we know that $\lim_{x\to-\infty} h(x) = 0$, then $a = 0$. In this case the solution is unique, the solution is also unique if we require for it the asymptotic: $h(x) \to a\sigma_3, a \neq 0$ when $x \to -\infty$ and a is some fixed constant. In both these cases one can write $\mathcal{R}_{(-)} = \mathcal{A}^{-1}$. However, we still have some difficulties. Indeed, for the geometric interpretations of the soliton equations hierarchies we must ensure that if $\mathcal{A}^{-1}g(x)$ satisfies $\lim_{x\to-\infty} \mathcal{A}^{-1}g(x) = a$ then $\lim_{x\to+\infty} \mathcal{A}^{-1}g(x) = a$ too. We can see that the above, generally speaking, is not true. But if this happens, it simply means that g is not in the image of $\mathcal{A}$. We also note that using solutions $\varphi(x)$ of the differential equation in (71), but this time defined by their behaviour at $+\infty$ instead at $-\infty$, we get another expression for the inverse. In order to distinguish these expressions we write $\mathcal{A}_-^{-1}$ and $\mathcal{A}_+^{-1}$. As a matter of fact, we have

$$\begin{aligned} h(x) &= \mathcal{A}_+^{-1} g(x) = \mathcal{R}_{(+)} g(x) \\ &= a\varphi\sigma_3\varphi^{-1}(x) + \frac{1}{2}\sum_{s=1}^{3} \varphi\sigma_s\varphi^{-1}(x) \int_{+\infty}^{x} B(g(y), \varphi\sigma_s\varphi^{-1}(y))\,\mathrm{d}y. \end{aligned} \tag{78}$$

Since φ and ψ are fundamental solutions of the same system of differential equations, there exists a non-degenerate constant matrix T such that $\psi = \varphi T$. The properties of the Killing form allow us to obtain that if $h(x)$ has the same asymptotic at $\pm\infty$, then

$$\mathcal{A}_+^{-1} g = \mathcal{A}_{(-)}^{-1} g(x) + \frac{1}{2}\sum_{s=1}^{3} \varphi\sigma_s\varphi^{-1}(x) \int_{-\infty}^{+\infty} B(g(y), \varphi\sigma_s\varphi^{-1}(y))\,\mathrm{d}y. \tag{79}$$

There is no contradiction here, because in the case when $g(x) = \mathcal{A}(h)$, where h is such that $(sw)\lim h(x) = a\sigma_3$, after integrating by parts and using the properties of the form B one can show that the integral in the right-hand side vanishes and $\mathcal{A}_-^{-1}g = \mathcal{A}_+^{-1}g$. The above again shows that inverting $\mathcal{A}$ one must be sure that one acts on functions that belong to the image of $\mathcal{A}$.

The difficulties we have encountered in inverting some operators are common in the geometric approaches to the soliton equations and in theory of the recursion operators, though they are simpler in the case when the potentials vanish at infinity. Even in that case, one usually must invert ∂_x (as in the case of the recursion operators for the HFeq) and of course the inverse is unique up to an additive constant. Different authors choose for the inverse one of the operators

$$\int_{-\infty}^{x}, \quad \int_{+\infty}^{x}, \quad \frac{1}{2}\left(\int_{-\infty}^{x} + \int_{+\infty}^{x}\right) \tag{80}$$

(the last being chosen in order to assure some symmetry properties) or simply write ∂_x^{-1}. Of course, there is no contradiction, since each time one uses recursion operators it is possible to prove that the integrands are total derivatives of polynomial functions on the potential and its derivatives (at least it is the case for the NLS hierarchy and the HF hierarchy it is so) or in other words, the integrands belong to the image of ∂_x. What happens in our case is similar. For example, solving

$$\frac{\mathrm{d}\mathbf{c}_1}{\mathrm{d}x} - K(\mathbf{S}) \times \mathbf{c}_1 = -K(\mathbf{S} \times \mathbf{b}_1) = K(\mathbf{S}_x(x)) \tag{81}$$

for $\mathbf{c}_1$ gives $K(\mathbf{S}) + \mathbf{h}$, where $\mathbf{h}$ satisfies the corresponding homogeneous equation. If we assume that the solution tends to $K(\mathbf{S}_0)$ when $|x| \to \infty$ then $\mathbf{c}_1 = K(\mathbf{S}(\mathbf{x}))$. The formula for the $\mathbf{c}_n$'s, see (41), shows that we can find the limit of $\mathbf{c}_n(x)$ as $|x| \to \infty$ and then $\mathbf{c}_n$ can be uniquely determined from the equation:

$$(\mathbf{c}_n)_x - K(\mathbf{S}) \times \mathbf{c}_n = -K(\mathbf{S} \times \mathbf{b}_n). \tag{82}$$

Therefore, $\mathcal{A}_\pm^{-1}(K(\mathbf{S} \times \mathbf{b}_n))$ is well defined.

Remark 3. The formula (41) avoids solving the differential equation for $\mathbf{c}_n$, but it gives $\mathbf{c}_n$ as a function of $\mathbf{b}_0, \mathbf{b}_1, \ldots, \mathbf{b}_{n-1}$. What we want however is an expression that depends only on $\mathbf{b}_n$.

Having in mind the properties of the operator $\mathcal{A}$, discussed in the above, the LLp chain system (39) can be written into the form

$$\begin{aligned} &\mathbf{S} \times \mathbf{b}_0 = 0 \\ &\mathbf{S} \times \mathbf{b}_{n+1} = -(\mathbf{b}_n)_x - \mathbf{S} \times K\mathcal{A}_\pm^{-1}K(\mathbf{S} \times \mathbf{b}_n) \\ &n = 0, 1, \ldots, \end{aligned} \tag{83}$$

where $\mathcal{A}_\pm^{-1}$ is used in the sense we discussed. The difficulties we mentioned can be overcome only after the thorough study of the spectral theory of the operator L in the Lax representation; in the present paper we limit ourselves to the geometric interpretation of the hierarchies.

For an arbitrary vector $\mathbf{q}$, the fact that $\mathbf{S} \in \mathbb{S}^2$ entails $\mathbf{q}^S = -\mathbf{S} \times (\mathbf{S} \times \mathbf{q})$, and we obtain

Proposition 6. *The solution of the LLp chain system can be put into the form:*

$$\begin{aligned} &\mathbf{b}_0 = \mathbf{S} \\ &\mathbf{b}_1 = \mathbf{S} \times \mathbf{S}_x \\ &\mathbf{b}_{n+1}^S = \Lambda_\pm^L\big(\mathbf{b}_n^S\big) \end{aligned} \tag{84}$$

$$\begin{aligned}&\mathbf{b}_{n+1} = \mathbf{b}^S_{n+1} + \mathbf{S}\int_{\pm\infty}^{x} \left\langle \mathbf{b}^S_{n+1}, \mathbf{S}_x \right\rangle \mathrm{d}x \\ &n = 1, 2, \ldots,\end{aligned} \tag{85}$$

where

$$\Lambda^L_{\pm}\mathbf{h} \equiv \Lambda_{\pm}(\mathbf{h}) + \mathbf{S} \times \left[\mathbf{S} \times K\mathcal{A}^{-1}_{\pm}K(\mathbf{S}\times\mathbf{h})\right] = \Lambda_{\pm}(\mathbf{h}) + \left[K\mathcal{A}^{-1}_{\pm}K(\mathbf{S}\times\mathbf{h})\right]^S \tag{86}$$

(of course, here we assume $\langle\mathbf{h}(x), \mathbf{S}(x)\rangle = 0$*).*

The operator $\Lambda^L_{\pm}$ is a recursion operator for the LLp chain system. When $K = 0$ it reduces to $\Lambda_{\pm}$—the recursion operator for the HF chain system, an advantage over the recursion operator obtained in [1], which reduces to $\Lambda^2_{\pm}$. Its flaw seems the presence of $\mathcal{A}^{-1}_{\pm}$, which cannot be expressed in terms of $\mathbf{S}(x)$ and its derivatives (in an explicit form). But since we have a nice recursion formula which gives the values $K\mathcal{A}^{-1}_{\pm}K(\mathbf{S}\times\mathbf{b}_n)$, this flaw is only apparent, actually one calculates the members of the LLp hierarchy at least as easily as one calculates them for the LLe hierarchy. There is however another thing that makes both operators quite different. The point is that, generally speaking, the LLp chain system is complex, and we cannot simply consider K real, because the system of interest (the LL equation) is obtained exactly when the entries of K are purely imaginary. But since the LL equation is real, it is desirable to have a real hierarchy. The expression for the generating operator, $\Lambda^L_{\pm}$, shows that it is complex too and hence the quantities $\mathbf{b}_n$ are complex. The same is deduced easily from the formula for $\mathbf{c}_n$, cf (41). The entries of the matrices $K^{(q)}$ are homogeneous polynomials of degree q in the entries of K, see [3], and hence for even q they are real and for odd q they are imaginary. This, together with

$$\mathbf{S}\times\mathbf{b}_{n+1} = -(\mathbf{b}_n)_x + \mathbf{S}\times K(\mathbf{c}_n), \tag{87}$$

show that, generally speaking, $\mathbf{b}_n$ have imaginary components. The imaginary terms vanish up to $n = 3$ when we start by $\mathbf{b}_0 = \mathbf{S}$, $\mathbf{c}_0 = 0$, because $KK^{(2)} = -i|j_1j_2j_3|\mathbf{1}$ and there are no terms proportional to i in the equation for $\mathbf{b}_3$. However, the same relation shows that in the expression for $\mathbf{b}_4$ there is an imaginary term, proportional to $\mathbf{S}\times\mathbf{S}_x$. What happens is that in the imaginary part of the hierarchy we have linear combinations of the same vector fields as in the real part. However, in the present context they appear together and is not clear how to separate them. The above also shows that one cannot expect that the recursion operator for the elliptic pencil will be the square of the recursion operator we have introduced. There must be some more subtle relation between them, a relation that will be revealed after the relation between the polynomial and elliptic pencils is better understood.

We are going now to show that for $\Lambda^L = \Lambda^L_{\pm}$ (we consider the two operators as equivalent) we have the usual geometric interpretation of the recursion operators, that is, Λ^L relates two Poisson structures and as a consequence its adjoint is a Nijenhuis tensor.

6. The geometric picture

6.1. Compatible Poisson–Lie tensors related to so(4)

As is well known, Poisson brackets over a manifold $\mathcal{M}$ can be introduced either by a symplectic form or by a Poisson tensor (Poisson structure). The symplectic case is classic, as to the Poisson structure, it is relatively new, though according to [31, 32] Lie's book, [15] contained the essentials of the theory of the Poisson manifolds (called in [15] 'function groups') and not only, as generally believed, only the so-called 'Poisson–Lie' structures on duals of Lie algebras. Also, as claimed by the same authors, in a book by Carathéodory [4] there has been a rather complete account of the theory based on even earlier work of Lie. In modern terms and notations the theory of the Poisson structures is described in [21, 14]. Below we remind some facts about these structures.

Let $T_m(\mathcal{M})$ and $T^*_m(\mathcal{M})$ be the tangent and the cotangent spaces at the point m of the manifold $\mathcal{M}$. A Poisson tensor is a smooth field of linear maps $m \to P_m \in \mathrm{End}(T^*_m(\mathcal{M}), T_m(\mathcal{M}))$ having the properties:

$$\begin{aligned} &\text{(i)} \quad P^*_m = -P_m \\ &\text{(ii)} \quad [P, P]_S = 0. \end{aligned} \tag{88}$$

Here $[\ ,\ ]_S$ denotes the so-called Schouten bracket of two tensor fields, which in the case of two (1,1) tensors P and Q have the form:

$$[P, Q]_S(\gamma_1, \gamma_2, \gamma_3) \equiv \left\langle Q L_{P\gamma_1}(\gamma_2), \gamma_3 \right\rangle + \left\langle P L_{Q\gamma_1}(\gamma_2), \gamma_3 \right\rangle + \mathrm{cycl}(1, 2, 3). \tag{89}$$

In the above expression $\gamma_1, \gamma_2, \gamma_3$ are 1-forms, (fields of covectors), $L_{P\gamma}$ is the Lie derivative corresponding to the vector field $P\gamma$ and $\langle\ ,\ \rangle$ means the canonical pairing between vector fields and covector fields (1-forms). The notation 'cycl (1,2,3)' in the right-hand side means that one must add to the expression all the terms obtained from the first two by cyclic permutation of the indices 1,2,3. The condition (ii) ensures the Jacobi identity and (i) guarantees the skew-symmetry of the Poisson bracket

$$\{f, g\} = \langle P\mathrm{d}f, \mathrm{d}g\rangle, \tag{90}$$

where f and g are functions over the manifold $\mathcal{M}$, or the 'generalized' Poisson bracket

$$\{\gamma_1, \gamma_2\} = -\mathrm{d}\langle P\gamma_1, \gamma_2\rangle, \tag{91}$$

where γ_1 and γ_1 are two closed 1-forms on $\mathcal{M}$.

Remark 4. When we write $P^* = -P$ we assume that the spaces $T^{**}_m(\mathcal{M})$ and $T_m(\mathcal{M})$ can be identified. This of course is always possible if the manifold $\mathcal{M}$ is finite dimensional, but if it is not, one has only $T_m(\mathcal{M}) \subset T^{**}_m(\mathcal{M})$. Then P^* is usually understood as the formal adjoint of P.

A manifold with a Poisson tensor on it is called a Poisson manifold.

There is a canonical way to equip the dual space $\mathfrak{G}^*$ of a Lie algebra $\mathfrak{G}$ with a Poisson structure, provided one can identify the vector spaces $\mathfrak{G}^{**}$ and $\mathfrak{G}$. This structure was discovered by Lie, [15] and was rediscovered later by several authors, [12, 29]. Suppose $\mu \in \mathfrak{G}^*$. Then one has

$$T_\mu(\mathfrak{G}^*) = \mathfrak{G}^*, \qquad T^*_\mu(\mathfrak{G}^*) = \mathfrak{G}^{**} = \mathfrak{G}. \tag{92}$$

The canonical Poisson structure over $\mathfrak{G}^*$ is defined by the following field of linear maps:

$$\mu \to L_\mu \in \mathrm{Hom}(\mathfrak{G}, \mathfrak{G}^*) : L_\mu(\xi) = -\mathrm{ad}^*_\xi \mu, \qquad \xi \in \mathfrak{G} \sim \mathfrak{G}^{**}. \tag{93}$$

The tensor L is called the Poisson–Lie tensor or the Kirillov tensor and the Poisson bracket defined by it is called the Poisson–Lie bracket, Kirillov bracket, Berezin bracket etc. We shall call the above tensor as the Poisson–Lie tensor and the corresponding bracket as the Poisson–Lie bracket.

If there exists symmetric non-degenerate bi-linear form $B(X, Y)$ over $\mathfrak{G}$, invariant with respect to the adjoint action of $\mathfrak{G}$, then one can identify in a canonical way $\mathfrak{G}^*$ and $\mathfrak{G}$, the adjoint and the coadjoint action and one can define Poisson structure over the algebra $\mathfrak{G}$:

$$q \to L_q \in \mathrm{End}(\mathfrak{G}) : L_q(\xi) = \mathrm{ad}_\xi q, \qquad \xi \in \mathfrak{G} \sim \mathfrak{G}^*. \tag{94}$$

Consider now the algebra $so(4)_J$. As a matter of fact, our interest the pencils of Lie algebras is motivated by the following simple observation—for arbitrary symmetric J the usual bracket and the bracket $[\,,\,]_J$ are compatible, that is, their sum is a Lie bracket. Then the corresponding Poisson–Lie tensors are compatible too. Taking into account the existence of the invariant bi-linear form $B_T(X, Y)$ we have

Theorem 1. *For each symmetric J there exists 2-parametric family of Poisson–Lie structures over the manifold $\mathcal{M} = so(4)$:*

$$\begin{aligned} &q \to P_q^{(a,b)} \in End(so(4), so(4)) \\ &P_q^{(a,b)}(\xi) = a[ad_\xi q] + b\big[ad_\xi^J q\big], \qquad \xi \in so(4) \sim so(4)^*. \end{aligned} \tag{95}$$

(a, b are the parameters and $X \mapsto ad_X^J$ is the adjoint representation of $so(4)_J$).

6.2. The Gel'fand–Fuchs cocycle and related Poisson structures, $so(4)$ case

The Poisson tensors used in the theory of the infinite-dimensional integrable equations are usually defined over manifolds consisting of functions defined on the real line and often include differentiation. It is known that there is an elegant way to define such tensors from tensors defined on finite-dimensional Lie algebras. Let $\mathfrak{G}_0[x]$ be the infinite-dimensional manifold of the smooth functions $f(x)$ defined on the real line, with values in the finite-dimensional Lie algebra $\mathfrak{G}$, tending fast enough to some constant value $f_0 \in \mathfrak{G}$ when $|x| \to \infty$. For obvious reasons we take the tangent space $T_f(\mathfrak{G}_0[x])$ at the point $f \in \mathfrak{G}_0[x]$ to be the vector space consisting of all Schwartz-type functions $\xi(x)$ on the line with values in $\mathfrak{G}$. We shall denote this space by $\mathfrak{G}[x]$. Clearly, if we define the Lie algebra operation point-wise, both $\mathfrak{G}_0[x]$ and $\mathfrak{G}[x]$ become Lie algebras and

$$[\mathfrak{G}[x], \mathfrak{G}_0[x]] \subset \mathfrak{G}[x]. \tag{96}$$

In order to introduce the Poisson structure over $\mathfrak{G}_0[x]$ we remind that if $\mathfrak{H}$ is a Lie algebra and γ is a fixed 2-cocycle of the trivial action of $\mathfrak{H}$ on the scalars, then one can define the following Poisson tensor over $\mathfrak{H}^*$:

$$\mu \to P_\mu : P_\mu(\xi) = \mathrm{ad}_\xi^* \mu + \gamma(\xi, .), \qquad \xi \in \mathfrak{H} \sim \mathfrak{H}^{**}. \tag{97}$$

This construction is related to the central extensions of $\mathfrak{H}$, see [11], but we shall not elaborate about this topic. As mentioned, the above construction is often applied in the theory of the soliton equations. In that case $\mathfrak{H}$ is chosen to be $\mathfrak{G}[x]$—the algebra of Schwartz-type functions on $\mathbb{R}$ or $\mathbb{C}$, taking values in a finite-dimensional Lie algebra $\mathfrak{G}$ (algebraic operations defined point-wise); see for example [24–27, 21, 6]. If $B_0(X, Y)$ is an invariant, non-degenerate bi-linear form on $\mathfrak{G}$ (in the case when the algebra $\mathfrak{G}$ is semisimple B_0 is usually the Killing form B, but in our case it will be different) there exists the following famous 2-cocycle of $\mathfrak{G}[x]$—the Gel'fand–Fuchs cocycle (sometimes called also Maurer–Cartan cocycle [6]):

$$\gamma(\xi, \eta) = c\int_{-\infty}^{+\infty} B_0(\partial_x \xi, \eta(x))\,\mathrm{d}x, \quad \xi, \eta \in \mathfrak{G}[x], \quad c = \text{constant}, \quad \partial_x \equiv \frac{\partial}{\partial x}. \tag{98}$$

Allowing some lack of rigor we say that we identify $\mathfrak{G}[x]$ and $\mathfrak{G}[x]^*$ using the invariant, non-degenerate bi-linear form on $\mathfrak{G}[x]$:

$$\langle\langle \xi, \eta\rangle\rangle = \int_{-\infty}^{+\infty} B_0(\xi(x), \eta(x))\,\mathrm{d}x, \qquad \xi, \eta \in \mathfrak{G}[x], \tag{99}$$

constructed from the symmetric invariant non-degenerate bi-linear form B_0 on $\mathfrak{G}$ ($\mathfrak{G}[x]$ and $\mathfrak{G}[x]^*$ cannot be identified in the proper sense so we must treat some elements as distributions). As a result we obtain the following Poisson tensor:

$$P_q(\xi) = [\xi, q] + c\partial_x \xi. \tag{100}$$

It can be verified that the above expression actually defines a Poisson tensor also in the case when the potential q belongs to the set of all smooth functions with values in $\mathfrak{G}$ tending fast enough to constant element from $\mathfrak{G}$ as $|x| \to \infty$, that is, to $\mathfrak{G}_0[x]$. Applying these constructions to $\mathfrak{G} = so(4)_{(a,b)}$, and using the invariant form B_T instead of B_0 in the definitions of the cocycle γ and of the inner product $\langle\langle X, Y\rangle\rangle$, we obtain

Theorem 2. *Over the manifold $so(4)_0[x]$ there exists a 3-parametric family of Poisson tensor fields:*

$$A \to P_A^{(a,b,c)} = -P_A^{(a,b)} + c\partial_x, \qquad A \in so(4)_0[x]. \tag{101}$$

The Poisson brackets are constructed in the following way. First, we identify any linear form α of the type

$$\alpha(Y) = \langle\langle X, Y\rangle\rangle \tag{102}$$

with the function $X(x)$, taking values in $so(4)$. (Of course, $X(x)$ can also be a distribution.) Then if $F[A]$ and $G[A]$ are two functionals we identify their differentials $\mathrm{d}F$ and $\mathrm{d}G$ with two $so(4)$-valued functions $X(x)$ and $Y(x)$ and construct the Poisson bracket

$$\begin{aligned}\{F, G\}_{(a,b,c)} &= \left\langle\!\left\langle -P_A^{(a,b)}(X) + c\partial_x X, Y\right\rangle\!\right\rangle \\ &= \int_{-\infty}^{+\infty} B_T(a[A(x), Y(x)] + b[A(x), X(x)]_J + c\partial_x X(x), Y(x))\,\mathrm{d}x.\end{aligned} \tag{103}$$

Upon our knowledge, the first time tensors of the type (101) have been used in [33] to describe the Hamiltonian properties of the chiral fields equations hierarchy. We shall use them now to describe the Hamiltonian properties of the LLp hierarchy.

6.3. The P–N structure for the LL hierarchy, obtained through polynomial pencil

We are going to apply some of the results outlined in the previous section for the manifold of potentials $\mathcal{M}_S^I$, but first some general geometric facts are needed. Suppose $\mathcal{M}$ is a manifold, endowed with two compatible Poisson structures. This means that on $\mathcal{M}$ are defined two Poisson tensors P and Q, and $P + Q$ also is a Poisson tensor. Then we have, see (89):

$$[P, P]_S = 0, \qquad [Q, Q]_S = 0, \qquad [P, Q]_S = 0, \tag{104}$$

where by $[\,,]_S$ is denoted the Schouten bracket. The existence of compatible Poisson structures (tensors) in one of the characteristic features of the integrable equations theory [8, 17, 24, 26, 27]. If in addition one of the tensors, for example Q, is invertible, then the pair (Q, N), $(N \equiv \Lambda^* \equiv Q^{-1} \circ P)$ endows the manifold on which P, Q are defined with the so-called Poisson–Nijenhuis structure (P–N structure) see [16]. N is a Nijenhuis tensor and this structure is responsible for the infinite Abelian algebra of symmetries one has for the soliton equations. For more details about the above structure we refer to [17], in what follows are going to define such a structure on $\mathcal{M}_S^I$. However, the tensors we shall use must be restricted, because they are defined on $so(4)_0[x]$. Also, the tensor we want to invert is not kernel free, so we cannot immediately perform the desired construction. Such situation occurs frequently and if in the compatible pair (P, Q) the Poisson tensor Q is not invertible, one can restrict it to some integral leaf of the distribution $m \mapsto \mathrm{im}(Q_m)$. In the finite-dimensional case, as is well known, on any such leaf Q allows restriction and the restricted tensor is invertible. Usually, one can do the same also in the infinite-dimensional case. The restriction of the other tensor (the tensor P), generally speaking, is more difficult, since restrictions of Poisson tensors are not always possible. However, there are sufficient conditions for this sort of restriction which for the soliton equation case are usually satisfied. These conditions are described in a theorem, proved in a general form in [20], see also [22, 23], and in a simpler version (which we present below) in [18, 19], where also a number of applications to the soliton equation theory are given.

Theorem 3. *Let $\mathcal{M}$ be Poisson manifold and $\mathcal{N} \subset \mathcal{M}$ be a submanifold. Let us denote by i the natural inclusion map of $\mathcal{N}$ into $\mathcal{M}$, by $\mathcal{X}_P^*(\mathcal{N})_m$ the subspace of covectors $\alpha \in T_m^*(\mathcal{M})$ such that*

$$P_m(\alpha) \in [di]_m(T_m(\mathcal{N})) = im([di]_m), \qquad m \in \mathcal{N}, \tag{105}$$

by $T^\perp(\mathcal{N})_m$—the set of all covectors at $m \in \mathcal{M}$ vanishing on the subspace $im([di]_m)$, $m \in \mathcal{N}$ (also called the annihilator of $im([di]_m)$ in $T_m^*(\mathcal{M})$). Suppose the following relations hold:

$$\mathcal{X}_P^*(\mathcal{N})_m + T^\perp(\mathcal{N})_m = T_m^*(\mathcal{M}), \qquad m \in \mathcal{N} \tag{106}$$

$$\mathcal{X}_P^*(\mathcal{N})_m \cap T^\perp(\mathcal{N})_m \subset ker(P_m), \qquad m \in \mathcal{N}. \tag{107}$$

Then there exists a unique Poisson tensor $\bar{P}$ on $\mathcal{N}$, i-related to P, that is, satisfying:

$$P_m = [di]_m \circ \bar{P}_m \circ [di]_m^*, \qquad m \in \mathcal{N}. \tag{108}$$

Now, looking at the LLp hierarchy of evolution equations, cf (14), (16), and using the definition of the tensor field $P_A^{(a,b,c)}$ one sees that one can cast them into the following equivalent forms:

$$A_t = P_A^{(\frac{1}{2},0,0)} B_{n+1}, \qquad A_t = P_A^{(0,-\frac{1}{2},1)} B_n \quad n = 0, 1, \ldots, \tag{109}$$

where as we have seen the two tensors are compatible. We cannot give immediately geometric interpretation of B_n but (109) draws our attention to the above tensors. For the sake of brevity we drop the complicated upper indices and denote the tensors in the above simply by P and Q, that is,

$$P_A^{(\frac{1}{2},0,0)} = Q_A = \tfrac{1}{2}\mathrm{ad}_A, \qquad P_A^{(0,-\frac{1}{2},1)} = P_A = \partial_x - \tfrac{1}{2}\mathrm{ad}_A^J. \tag{110}$$

The tensor Q is familiar, in vector representation it defines the so-called first Hamiltonian structure on $\mathcal{M}_S$, the tensor P is more 'exotic'.

Now we want to restrict these tensors on the space of potentials

$$\mathcal{M}_S^I = \{A = \{\mathbf{S}\}_I,\ \mathbf{S}(x) \in \mathcal{M}_S\} \sim \mathcal{M}_S. \tag{111}$$

The first tensor (Q) is immediately restricted, its form remains the same, so we denote the restriction by the same letter. It remains to restrict the second tensor (P). In order to check the conditions that appear in theorem 3 we calculate

$$\begin{aligned} T_A\big(\mathcal{M}_S^I\big) &= \{X : X = [A, Y],\, Y \in \mathfrak{G}_I[x]\} \\ T_A^\perp\big(\mathcal{M}_S^I\big) &= \{fA + Y^* : f \in \mathcal{S}(\mathbb{R}),\, Y^* \in \mathfrak{G}_{II}[x]\}, \end{aligned} \tag{112}$$

where $\mathcal{S}(\mathbb{R})$ denotes the space of the Schwartz functions on the line. As explained, we identify the tangent and cotangent vectors using the form $\langle\langle\,,\rangle\rangle$, but we write star as superscript when we are dealing with covectors in order to make the geometric meaning clearer. Next

$$\mathcal{X}_P^*\big(\mathcal{M}_S^I\big)_A = \{X^* + Y^* : X^* \in \mathfrak{G}_I[x],\, Y^* \in \mathfrak{G}_{II}[x]\}, \tag{113}$$

where X^* and Y^* satisfy

$$\begin{aligned} &(X^*)_x - \tfrac{1}{2}[A, \alpha(Y^*)] \in T_A\big(\mathcal{M}_S^I\big) \\ &(Y^*)_x - \tfrac{1}{2}[\alpha(A), Y^*] = -\tfrac{1}{2}\alpha([A, X^*]). \end{aligned} \tag{114}$$

The first of these relations implies that

$$B_T(X^*_x(x), A(x)) = 0. \tag{115}$$

The second one is equivalent to the pair of vector equations:

$$\mathbf{c}_x - K(\mathbf{S}) \times \mathbf{c} = -K(\mathbf{S} \times \mathbf{b}) \tag{116}$$

for $X^* = \{\mathbf{b}\}_I$, $C^* = \{\mathbf{c}\}_{II}$. As one can see the operator $\partial_x - \frac{1}{2}\mathrm{ad}_{\alpha(A)}$ is the matrix form of the operator $\mathcal{A}$, introduced in (65), so we shall denote it by the same letter. Then the equation from (114) reads

$$\mathcal{A}(Y^*) = -\tfrac{1}{2}\alpha([A, X^*]). \tag{117}$$

The domain D of the operator $\mathcal{A}$ is the set of the smooth functions $Y^*(x)$ taking values in $\mathfrak{G}_{II}$ and tending fast enough to some constants $\{(0, 0, a)\}_{II}$. In this case, as we have seen, the operators $\mathcal{A}_\pm^{-1}$ are well defined on the image of $\mathcal{A}$ and give the same result. From the fact that $\mathcal{A}$ has trivial kernel when restricted to $\mathfrak{G}_{II}[x]$ we get

$$\left(\mathcal{X}^*_P\left(\mathcal{M}^l_S\right)_A\right) \cap T^\perp_A\left(\mathcal{M}^l_S\right) = \{0\}. \tag{118}$$

Finally, if $X^* + Y^*$ is a covector from $T^*_A(so(4)_0[x])$, making use of (36) we obtain

$$\begin{aligned} X^* + Y^* &= Z_1^* + Z_2^* \\ Z_1^* &= X^* + \frac{1}{8}A\int_{\pm\infty}^x B_T(A(y), X_y(y))\,\mathrm{d}y - \frac{1}{2}\mathcal{A}_\pm^{-1}(\alpha([A, X^*])) \\ Z_2^* &= Y^* - \frac{1}{8}A\int_{\pm\infty}^x B_T(A(y), X_y(y))\,\mathrm{d}y + \frac{1}{2}\mathcal{A}_\pm^{-1}(\alpha([A, X^*])). \end{aligned} \tag{119}$$

Since $Z_1^* \in \mathcal{X}^*_P\left(\mathcal{M}^l_S\right)_A$, $Z_2^* \in T^\perp_A\left(\mathcal{M}^l_S\right)$ the above shows that

$$\left(\mathcal{X}^*_P\left(\mathcal{M}^l_S\right)_A\right) + T^\perp_A\left(\mathcal{M}^l_S\right) = T^*_A(so(4)_0[x]). \tag{120}$$

Therefore, the requirement of theorem 3 are fulfilled and P allows restriction. According to the prescriptions of this theorem, for $X^* \in T^*_A\left(\mathcal{M}^l_S\right)$ one must take first $i^*(X^*)$, where i is the canonical inclusion map, then represent it as a sum $Y_1^* + Y_2^*$, where $Y_1^* \in \mathcal{X}^*_P\left(\mathcal{M}^l_S\right)_A$, $Y_2^* \in T^\perp_A\left(\mathcal{M}^l_S\right)$, and the restricted tensor will satisfy $\bar{P}_A(X^*) = P_A(Y_1^*)$. In our case $i^*(X^*) = X^*$ so we get

$$\begin{aligned} \bar{P}_A(X^*) &= \partial_x(X^*) + \frac{1}{8}A_x\int_{\pm\infty}^x B_T(X^*_y(y), A(y))\,\mathrm{d}y \\ &\quad + \frac{1}{8}B_T(X^*_x, A)A - \frac{1}{4}\left[A, \left(\alpha\circ\mathcal{A}_\pm^{-1}\circ\alpha\right)([A, X^*])\right] \\ &= \pi(\partial_x(X^*)) + \frac{1}{8}A_x\int_{\pm\infty}^x B_T(X^*_y(y), A(y))\,\mathrm{d}y - \frac{1}{4}\left[A, \mathcal{A}_\pm^{-1}(\alpha([A, X^*])\right], \end{aligned} \tag{121}$$

where π denotes the projection in $so(4)$ onto the subspace, orthogonal to A. Also, since $B_T(X^*(x), A(x)) = 0$, one has

$$B_T(X_x(x), A(x)) = -B_T(X(x), A_x(x)). \tag{122}$$

Finally we obtain

Proposition 7. *On the manifold of potentials $\mathcal{M}^l_S$ there exists a restriction $\bar{P}$ of the Poisson tensor P, having the form:*

$$\bar{P}_A(X^*) = \pi(\partial_x(X^*)) - \frac{1}{8}A_x\int_{\pm\infty}^x B_T(X^*(y), A_y(y))\,\mathrm{d}y - \frac{1}{4}\left[A, \left(\alpha\circ\mathcal{A}_\pm^{-1}\circ\alpha\right)([A, X^*])\right], \tag{123}$$

where $X^* \in T^*_A\left(\mathcal{M}^l_S\right)$.

Since on $T_A\big(\mathcal{M}_S^l\big)$ the operator Q_A is invertible, we are able to calculate $\Lambda_\pm^L = Q_A^{-1}\circ\bar{P}_A$:

$$\begin{aligned}\Lambda_\pm^L(X^*) &= -\frac{1}{2}[A,\partial_x(X^*)] + \frac{1}{16}[A,A_x]\int_{\pm\infty}^x B_T(X^*(y),A_y)\,\mathrm{d}y\\ &\quad -\frac{1}{2}\big(\pi\circ\alpha\circ\mathcal{A}_\pm^{-1}\circ\alpha\circ\mathrm{ad}_A\big)(X^*)\\ &= \Lambda_\pm(X^*) - \frac{1}{2}\big(\pi\circ\alpha\circ\mathcal{A}_\pm^{-1}\circ\alpha\circ\mathrm{ad}_A\big)(X^*).\end{aligned} \tag{124}$$

(Of course, in the above we assume that X^* satisfies $B_T(X^*(x),A(x)) = 0$). Some explications are needed for the last formula. If we calculate the above in vector representation we shall get that the term

$$\Psi_\pm(X^*) = -\frac{1}{2}[A,\partial_x(X^*)] + \frac{1}{16}[A,A_x]\int_{\pm\infty}^x B_T(X^*(y),A_y)\,\mathrm{d}y, \tag{125}$$

for $X^* = \{\mathbf{b}\}_I$ is equal to $\{\Lambda_\pm\mathbf{b}\}_I$, and the whole expression in (124) equals

$$\Lambda_\pm^L(X^*) = \big\{\Lambda_\pm(\mathbf{b}) + \big[\big(K\mathcal{A}_\pm^{-1}K\big)(\mathbf{S}\times\mathbf{b})\big]^S\big\}_I, \tag{126}$$

which explains why we denote the operator $Q_A^{-1}\circ\bar{P}_A$ by $\Lambda_\pm^L$, and the operator $\Psi_\pm$ in (125) by $\Lambda_\pm$—they are just the recursion operators for LLp and HF in vector representation, cf (86) and (46). As we have seen, the action of $\mathcal{A}_+^{-1}$ and $\mathcal{A}_+^{-1}$ gives the same result, so below we write Λ^L instead of $\Lambda_\pm^L$. The general theory of compatible Poisson tensors then yields

Theorem 4. *The pair $(Q, N^L = (\Lambda^L)^*)$ endows the manifold of potentials $\mathcal{M}_S^I$ (or $\mathcal{M}_S^l$) with a P–N structure.*

6.4. *Hamiltonian properties of the LLp hierarchy*

Let us consider again the equations forming the LLp hierarchy, (14), (16). According to (109) we can write them as

$$A_t = Q_A(B_{n+1}) = P_A(B_n), \qquad n = 0,1,\ldots, \tag{127}$$

where $B_n = F_n + G_n$ ($F_n\in\mathfrak{G}_I$, $G_n\in\mathfrak{G}_{II}$) satisfy the chain system (15), or equivalently (37), which for our convenience we write again

$$\begin{aligned}&[A,F_0] = 0\\ &\tfrac{1}{2}[A,F_{n+1}] = (F_n)_x - \tfrac{1}{2}[A,\alpha(G_n)]\\ &(G_n)_x - \tfrac{1}{2}[\alpha(A),G_n] + \tfrac{1}{2}\alpha([A,F_n]) = 0.\end{aligned} \tag{128}$$

We want to give geometric interpretation to these relations. The third equation is equivalent to

$$G_n = -\tfrac{1}{2}\mathcal{A}_\pm(\alpha([A,\pi(F_n)]).$$

As to the second equation, first it shows that $(F_n)_x \in T_A\big(\mathcal{M}_S^l\big)$ and therefore $B_T(A(x),(F_n(x))_x) = 0$. Next, since $F_n = \pi(F_n) + B_T(A,F_n)A$ and

$$B_T(A,F_n)(x) = \int_{\pm\infty}^x \partial_y B_T(A(y),F_n(y))\,\mathrm{d}y = \int_{\pm\infty}^x B_T(A_y(y),F_n(y))\,\mathrm{d}y,$$

it is easily seen that it can be written as

$$Q_A(\pi(F_{n+1})) = \bar{P}_A(\pi(F_n)). \tag{129}$$

But then the hierarchy (14), (16) can also be cast into the form:

$$A_t = Q_A(\pi(F_{n+1})), \qquad A_t = \bar{P}_A(\pi(F_n)). \tag{130}$$

We therefore can interpret $\pi(F_n)$ as 1-forms on $\mathcal{M}_S^J$ and the above equations as Hamiltonian equations, provided $\pi(F_n) = \mathrm{d}H_n$, where H_n are the Hamiltonians for these equations. The theory of the P–N manifolds shows that if on a P–N manifold the 1-form β is fundamental (corresponds to fundamental field of the structure) then β is closed and if β, $N^*\beta$ are closed, then $(N^*)^k\beta$, $k \geqslant 2$ are closed too. In our case this means that all $\pi(F_n)$ are closed forms. Indeed, $\pi(F_1)$ is closed and fundamental (it can be shown exactly as was done in [33] for the $O(3)$ chiral fields system). The form $\pi(F_2)$ is even exact, since it corresponds to the LL equation and it is Hamiltonian. So the equations (130), starting from the second one (the LLeq) are Hamiltonian and even bi-Hamiltonian in a generalized sense. As for the proof that there exist well-defined functionals H_n, such that $\pi(F_n) = \mathrm{d}H_n$ (in general one has it starting from some n_0), this fact usually follows from some other considerations. For example, one gets this result for the HF equation and NLS equation from the spectral theory for the corresponding auxiliary liner problem and the spectral theory of the recursion operators. It is well known that for NLSeq and HFeq the Hamiltonians can be obtained through the recursion operators [10], though their locality is not immediately seen and needs to be proved [9]. We shall assume that H_n exist for $n \geqslant 2$. If so, the equations (130), together with (129), show that the nonlinear evolution equation hierarchy, related to the polynomial pencil is bi-Hamiltonian and is related to the P–N structure described in theorem 4. From the general theory of the P–N manifolds we get

Corollary 3. *The nonlinear equations from the hierarchy (16) (from the hierarchy (17)), are Hamiltonian, and their Hamiltonians are in involution.*

Finally, we note that the above corollary can be obtained also directly. Indeed, if we denote by $\{H_n, H_m\}_Q$, $\{H_n, H_m\}_P$ the Poisson brackets defined by Q and $\bar{P}$ respectively, we can write the equalities:

$$\begin{aligned}\{H_n, H_m\}_Q &\equiv \frac{1}{2}\langle\langle [A, \pi(F_n)], \pi(F_m)\rangle\rangle \\ &= \langle\langle \bar{P}(\pi(F_{n-1}))], \pi(F_m)\rangle\rangle \equiv \{H_{n-1}, H_m\}_P \end{aligned} \tag{131}$$

$$\begin{aligned}\frac{1}{2}\langle\langle [A, \pi(F_n)], \pi(F_m)\rangle\rangle &= \frac{1}{2}\langle\langle [A, B_n], B_m\rangle\rangle \\ &= \frac{1}{2}\int_{-\infty}^{+\infty} B_T([A(x), B_n(x)], B_m(x))\,\mathrm{d}x \end{aligned} \tag{132}$$

$$\begin{aligned}\{H_{n-1}, H_m\}_P &\equiv \langle\langle \bar{P}(\pi(F_{n-1}))], \pi(F_m)\rangle\rangle \\ &= \int_{-\infty}^{+\infty} B_T\left(\partial_x B_{n-1}(x) - \frac{1}{2}[A(x), B_{n-1}(x)]_J, B_m(x)\right)\mathrm{d}x. \end{aligned} \tag{133}$$

Now we implement a technique described in [8] and frequently used in similar proofs. Let us suppose for definitiveness that $n > m$. Using that ad_X^J is skew-symmetric with respect to the bi-linear form B_T, integrating by parts and taking into account the properties of the matrices B_n, see (45), we get that

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q. \tag{134}$$

Applying this identity sufficiently many times we arrive at the equation of the type

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q = \cdots = \{H_k, H_k\}_Q \tag{135}$$

if $n - m$ is even, or to the equation of the type

$$\{H_n, H_m\}_Q = \{H_{n-1}, H_{m+1}\}_Q = \cdots = \{H_k, H_{k-1}\}_Q = \{H_{k-1}, H_k\}_Q \tag{136}$$

if $n - m$ is odd. In both cases we conclude that $\{H_n, H_m\}_Q = 0$. From the proof it follows also that $\{H_n, H_m\}_P = 0$. This is exactly what we wanted to show.

7. Conclusion

We have considered the chain system resulting from a polynomial pencil of Lax pairs containing a Lax pair for the Landau–Lifshitz equation, the set of integrable equations related to that pencil and the corresponding recursion operators $\Lambda^L_\pm$. We have seen that these equations are bi-Hamiltonian, and that the recursion operators relate two compatible Poisson structures, that is, $\Lambda^L_\pm$ possess the geometric meaning the recursion operators usually have. We are able to claim that $\Lambda^L_\pm$ are recursion operators for the LL equation for the same reason as the operators found by Barouch *et al* in [1]. The above-mentioned facts serve to further motivate the interesting question of whether there is equivalence between the two pencils containing L-A pairs for the Landau–Lifshitz equation—the elliptic pencil and the polynomial pencil. We believe that this will attract the interest of the specialists.

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